Imran NASIR, PhD student
E-mail: imrannasir.07435@iba.khi.edu.pk
Department of Mathematical Sciences, School of Mathematics and Computer Science
Institute of Business Administration Karachi, Pakistan
Assistant Professor Muhammad SHERAZ, PhD
E-mail: msheraz@iba.edu.pk
Department of Mathematical Sciences, School of Mathematics and Computer Science
Institute of Business Administration Karachi, Pakistan
Department of Financial Mathematics, Fraunhofer ITWM Kaiserslautern, Germany
Associate Professor Silvia DEDU, PhD
E-mail: silvia.dedu@csie.ase.ro
Department of Applied Mathematics
The Bucharest University of Economic Studies
"Costin C. Kiriţescu" National Institute of Economic Research

MIXTURE MODELS AND MODELLING VOLATILITY OF RETURNS – A STUDY ON GAUSSIAN AND HETEROGENEOUS HEAVY TAIL MIXTURES

Abstract. *This paper proposes several mixture models for describing return series of Pakistan Stock Exchange Index (PSX-100) and Pakistani Rupee (PKR) exchange rate against the US dollar. We implement Gaussian mixtures, heterogeneous mixtures and Heavy-tailed models of return distributions. The following models have been employed for this purpose: (i) Gaussian mixture up to three components, (ii) heterogeneous mixtures such as Normal-Logistic, Normal-Laplace, Logistic-Laplace, Normal-Normal inverse Gaussian (NIG), Logistic-NIG, Laplace-NIG, and Logistic-Skewed Generalized T distribution, and (iii) non-Gaussian heavy-tailed returns distributions such as Student T, Skewed-Generalized T, Logistic, Laplace, Generalized Hyperbolic and its sub-family such as NIG and Variance Gamma distribution. In addition, GARCH models and Value-at-Risk measure are used to assess the contribution of leverage effect, volatility clustering and asymmetric nature of the underlying returns series. Finally, we investigate the detrended correlation coefficient analysis to evaluate the integration level of the two markets and their role in the country's economy.*

Keywords*: non-Gaussian modelling; ARCH/GARCH; decentralized correlation coefficient analysis; average cross-correlation analysis; exchange rate; stock markets.*

JEL Classification : E00, C22, C52, C53

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1. Introduction

Classical portfolio theory by Markowitz (1952), Black-Scholes model (1973) and other financial literature assume that returns series are normally distributed. Empirical studies suggest that in emerging markets, returns are non-Gaussian, with high-kurtosis, fat-tails and non-persistent volatilities. Therefore, it is crucial to comprehend the validity and implications of these assumptions for financial markets on a case-to-case basis. As an emerging market, the Pakistani equity and currency market offers an opportunity to understand the behaviour and cross-linkages between two types of emerging markets. In literature, small coverage has been given to the statistical properties of the Pakistan Stock Exchange Index (PSX-100) and Pakistani Rupee (PKR) exchange rate against the US dollar. In international stock markets, an increasing trend of equity investments explores the demand and supply of foreign currencies. Therefore, it is vital to investigate the relationship between the stock and currency returns. This study considers PSX-100 (formerly known as KSE 100) returns and PKR/USD exchange rate returns. In 2016, PSX-100 was declared an emerging Asian market. Pakistan's qualification for the Morgan Stanley Capital International (MSCI) Emerging market index in 2017 resulted in a record high of 49,876 base points, which is unbroken to this day. On the other hand, the exchange rate of Pakistan has experienced significant depreciation of more than 700% and increasingly wild swings in the last three decades. Both Pakistani equity and currency markets are emerging and characterized by high and persistent volatilities.

In the past, several studies have been accomplished to examine the volatility dynamics and other statistical characteristics of PSX-100 and PKR exchange rates in the US dollar. For example, see, Abu and Lucjan (2014), as well as Sheraz and Imran (2021). The novel approach in our paper is represented by the examination of fat-tailed returns behaviours, volatility dynamics, and employment of the mixture models of returns distributions for the stock and currency market of Pakistan. The literature on the volatility dynamics of PSX-100 and the Pakistan currency market is limited. To the best of our knowledge, no studies have been conducted on identifying the fat-tailed properties, employment of the mixtures, and heterogonous mixture distribution models for the two underlying markets. This paper aims to fill the existing research gap on the PSX-100 and PKR-USD exchange rates.

Mandelbrot (1963) proposed a fat-tail distribution model in finance. Consequently, Lévy's stochastic models proposed in the early 1980s explore different stylized features of financial assets. These models have many appealing properties in financial economics and are extensively used in the literature. See, for example, applications in stock and currency returns by Barndorff-Nielsen (1997), Eberlein & Keller (1995), Schoutens (2003) and Cont and Tankov (2004). In the literature, mixture models of return distributions have outperformed as an alternative. For this purpose, Gaussian and heterogeneous mixtures are considered effective in modelling stock and currency returns. Some recent studies on mixture modelling can be found in Corlu and Corlu (2015) and also in Massing and Ramos (2021).

Keeping in view the above discussion, the original contribution of the present paper is four-fold: this study aims to identify the fat-tailed characteristics of PSX-100 and PKR-USD by using (i) Logistic, Laplace, Student-T, Skewed Generalized T, Generalized Hyperbolic, Normal inverse Gaussian (NIG) and Variance Gamma distributions (ii) we propose Gaussian mixtures and heterogeneous mixtures of underlying distributions to model and compare the performance of each of the models (iii) we test the presence of volatility clustering and leverage effects by using GARCH-type modelling of five different GARCH models such as SGARCH by Bollerslev (1986), GJR-GARCH due to Glosten et al. (1993), EGARCH by Nelson (1991), APARCH by Ding et al. (1993) and CSGARCH by Lee and Engle (1999), each with five conditional distributions. (iv) Finally, we present Value-at-Risk (VaR) and discussion on detrended cross-correlation. In recent studies, we can find systematic literature reviews on GARCH models and applications. See, for example, Dhanaiah and Prasad (2017), Hussain et al. (2019), Bhowmik and Wang (2020).

The paper is organized as follows sections. Section 2 describes the data. In Section 3, we present the methodology employed. Section 4 provides empirical results, and Section 5 presents the conclusions.

2. Data

The dataset containing the PSX-100 index, and Pakistani Rupee (PKR), exchange rate against the US Dollar, is extracted from Yahoo Finance. Figure 1 represents the daily prices of the PSX-100 index and PKR-USD exchange rate, which suggests the non-stationary behaviour of these variables. It also indicates that the volatility of the stock market is higher than that of the currency market.

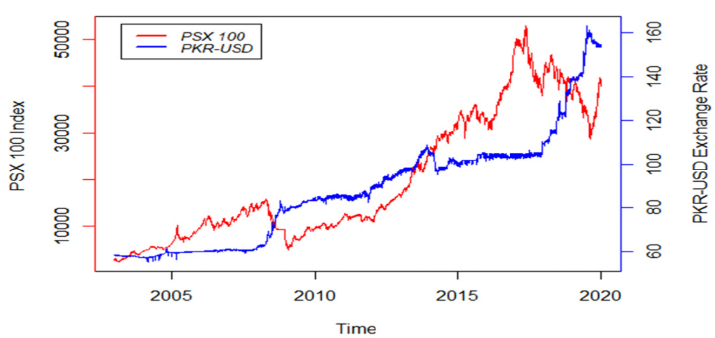

Figure 1. Daily closing prices of Pakistan Stock Exchange index (PSX-100) and Pakistani Rupee exchange rate against US dollar (PKR-USD) from Jan. 1, 2003, to Jan. 1, 2020

For any $t \in [0, T]$, let x_t denotes the daily closing prices data. The logarithmic returns of daily closing prices are given by:

$$R_t = \log\left(\frac{x_t}{x_{t-1}}\right) \quad (1)$$

The returns are negatively and positively skewed for stock and currency, respectively. More important, high kurtosis values of 5.873 and 9.309 are observed in daily closing returns of PSX-100 and PKR-USD returns series, respectively. The Jarque–Bera (JB) test rejects the hypothesis which states that data follows the normal distribution. The Augmented Dickey-Fuller (ADF) test shows small p −values that indicate stationarity. According to Ljung-Box (LB) test, no serial correlation exists in either time series.

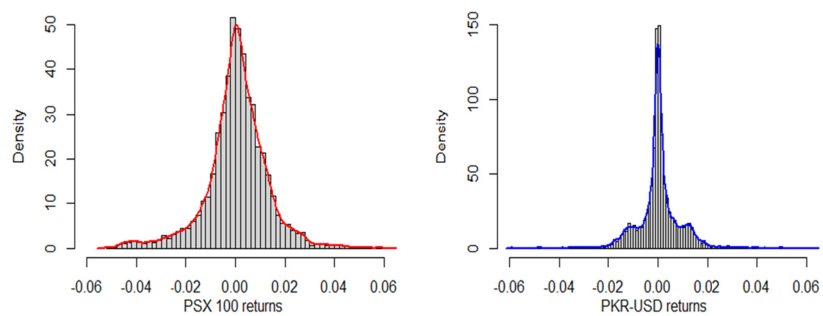

Figure 2. Probability Density of PSX-100 returns (left) for daily closing prices Probability Density for PKR-USD returns (right) for daily closing prices

3. Methods

3.1. Return Distributions, Gaussian Mixture and Heterogeneous Mixture Models

We consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let X be a continuous random variable, which models the logarithmic returns of the underlying asset. We denote by $g$the probability density function (PDF) and by $G$the cumulative distribution function (CDF) of the random variable X. Since the introduction of the heavy-tail model by Mandelbrot (1963), the use of these models for stock and currency returns provides the best results. Therefore, a choice of good distribution is vital for modeling returns and corresponding risk assessment. We consider return distributions which include Logistic, Laplace, Student-T, skewed-Generalized T, NIG, Generalized Hyperbolic and Variance Gamma distribution. On the other hand, the mixture of Gaussian distribution is a widely used technique to model stock and currency returns. These mixture models are attractive for several reasons. For example, (i) the model applies equally well to univariate and multivariate settings, and (ii) the mixtures can achieve outstanding flexibility with only a few components. (iii) it has the flexibility to adjust its component weights, means and variances. Therefore, we can approximate various continuous distributions using these mixtures. Additionally, mixture models are also effective in capturing the non-Gaussian aspect of returns. Suppose w_i represents the weight and $g_i(x|\,\theta_i)$

represents the probability density function corresponding to the i^{th} component of a mixture model. A finite mixture model can be defined as:

$$g(x|\{\theta_i\}) = \sum_{i=1}^{k} w_i g_i(x|\,\theta_i) \tag{2}$$

where w_i denotes the relative weight of the i^{th} component, with $0 \leq w_i \leq 1$ and $\sum_{i=1}^{k} w_i = 1$ and θ_i denotes the component-specific parameter set. In Gaussian mixtures, all components follow the Gaussian distribution. In this study, we implement Gaussian mixture model of two, and respectively three components. A Gaussian mixture model defined by a k number of components can be represented as follows:

$$g(x|\{\theta_i\}) = \sum_{i=1}^{k} w_i \mathcal{N}(x|\mu_i, \Sigma_i) \tag{3}$$

where $\mathcal{N}(x|\mu_i, \Sigma_i)$ is referred to as Gaussian component of the mixture model with specific values of mean μ_i and covariance Σ_i. The properties of the Gaussian mixture can be defined by the parameters $\boldsymbol{w} = (w_1, w_2, \cdots, w_k)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_k)$, and $\boldsymbol{\Sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_k)$. Clark (1973) introduced heterogeneous mixtures composed by a combination of different probability distributions. Due to the unique nature of components, these mixtures can capture a wide variety of returns and offer more flexibility comparative to Gaussian mixtures. The heterogeneous mixtures can be mathematically described by equation (2), where $g_i(X|\,\theta_i)$ denotes the component probability density function. For heterogeneous mixtures we use Normal, Logistic, Laplace, NIG and skewed-generalized T distributions to construct two-component heterogeneous mixtures.

3.2. Estimation Method and Model Selection Criteria

The maximum likelihood method is used to fit the underlying univariate distributions. The maximum-likelihood estimator is the parameter set that maximizes the likelihood function. Suppose $x_1, x_2, \cdots, x_n$ represent a set of observations corresponding to the random variable *X* and $\Theta = (\theta_1, \theta_2, \cdots, \theta_k)$ is the parameter of distribution $g(x|\,\Theta)$ followed by *X*. The general idea is to select estimators from the parameter set Θ that maximize the likelihood function of the sample observations. In other words, for estimated values of Θ the observed sample is also the most likely sample. The log likelihood function is given by:

$$\ln L(\Theta) = \sum_{i=1}^{n} \ln g(x_i|\,\Theta) \tag{4}$$

For mixture models, we use the Expectation Maximization (EM) algorithm, which is a general method to deal with the iterative computation of maximum likelihood estimation. The EM method works with initial parameter estimates and then iterates through two steps (i) the Expectation step and (ii) the Maximization step. The former step assumes fixed parameter estimates and computes the expected values of the latent variables in the model.

The maximization step updates the previous parameter estimates that maximize the likelihood function.

Model selection is one of the crucial steps of statistical modelling. Various Goodness-of-fit criteria are used to find the best-fit model. We use probabilistic statistical measures to quantify the performance of the underlying models. Multiple penalized criteria assess the quality of the model while incorporating samples size bias and complexity of the model at the same time. However, each model has its advantages and disadvantages. For this study, we use model selection methods, involving Akaike information criterion with its modifications (AIC, CAIC, AICC), Bayesian information criterion (BIC) and Hannan & Quinn criterion (HQC).

3.3. GARCH Modelling, Risk Measures and Detrended Cross-Correlation Coefficient Analysis

One of the crucial characteristics of financial market volatility is the time-varying nature of asset returns fluctuations. Generalized Autoregressive Conditionally Heteroscedastic models, abbreviated as GARCH, are the most popular and flexible to capture the volatility-clustering and other characteristics of the market volatility. This study considers the standard GARCH model and asymmetric GARCH models to estimate the volatilities of two data sets. Let R_t represents logarithmic returns, and $R_t = \mu + e_t$, also first two conditional moments exist, and $E(e_t|e_u) = 0$. Suppose σ_t^2 and e_t^2 represents conditional variance and squared error, and $e_t = \sigma_t Z_t$, and $Z_t \sim N(0,1)$. We employ SGARCH (Bollerslev 1986), APARCH (Ding et al. 1993), GJR-GARCH (Glosten et al. 1993), EGARCH (Nelson 1991) and CSGARCH (Lee and Engle 1999). See Appendix A for details. We also evaluate the out of sample forecasting performances of these models.

In modern financial literature, the most frequently used risk measure is Value-at-Risk (VaR) and it is used to evaluate the losses concerning market dynamics. The quantification of risk may provide valuable insight for making investment related decisions. See details in Jorion (2001), Sheraz and Dedu (2020). Finally, we find the linkage between PSX-100 and PKR exchange rates by employing the concept of the Detrended Cross-Correlation Coefficient (DCCA) proposed by Zebende. The DCCA measures the long-range cross-correlation between two series. The method is used to estimate the long-run cross-correlation between equity and currency markets. DCCA provide a dimensionless measure to compare the degree of co-movements for the underlying assets. DCCA is defined as the ratio between detrended covariance and the product of two detrended standard deviations. The cross-correlation coefficient was introduced to quantify the level of cross-correlation between nonstationary timeseries. Mathematically it is given by:

$$\rho_{X,Y}(n) = \frac{\sigma_{X,Y}^2(n)}{\sigma_{X,X}(n) \cdot \sigma_{Y,Y}(n)} \tag{5}$$

The value of $\rho_{X,Y}(n)$ ranges between -1 $\leq \rho_{X,Y}(n) \leq$ 1. A value of $\rho_{X,Y}(n) = 0$ means there is no cross-correlation.

4. Empirical Results

4.1 Models of Returns Distribution

In this section, we present empirical results of the PSX-100 and PKR returns series. We employed Logistic, Laplace, Student-T, Skew. Gen. T, NIG, Gen. Hyperbolic and Variance Gamma distribution for both series of returns.

We employed the underlying fat-tailed distributions for both PSX-100 and PKR-USD returns. Our computed ranks for goodness-of-fit of distributions are based on AIC, BIC, CAIC, AICC and HQC, and log-likelihood. A model with a minimum average rank for a selected variable is considered the best-fit model. Under this approach, Normal Inverse Gaussian (NIG) serves as the best-fit distribution for both PSX-100 and PKR-USD, which indicates the fat-tailed and leptokurtic behavior of these markets. The estimated parameters of heavy-tailed distributions reflect negatively skewed and fat-tailed behavior of PSX-100. On the other hand, a similar fat-tailed trend is observed in the case of PKR-USD apart from positive skewness. We extend our study to employ the Gaussian mixture distributions for both return series. A three component Gaussian mixture model is evaluated under the minimum average rank approach discussed earlier, which suggests that a three-component mixture is the best fit. Also, we use the Kolmogorov-Smirnov (K-S) test to compare the suitability of underlying models. The smaller value of Kolmogorov-Smirnov statistics indicated the best fit. Similarly, a large p-value shows the best-fit model. We have observed for three-component Gaussian mixtures, the p-values of both PSX-100 returns (0.261) and PKR-USD returns (0.101) are higher than in a two-component mixture model.

Table 1. Estimated parameters for best-fit NIG, Gaussian mixtures and heterogeneous mixture models implemented on Pakistan Stock Exchange (PSX-100) returns

Model Type	Best-Fit Model	Parameters		
Heavy-tailed distributions	NIG	$\mu = 0.00146$		
		$\delta = 0.0112$		
		$\alpha = 57.47$		
		$\beta = -5.96$		
Gaussian mixture	Three components	$w_1 = 0.03$	$= -0.031$	$\sigma_1 = 0.010$
		$w_2 = 0.65$	$= 0.0010$	$\sigma_2 = 0.007$
		$w_3 = 0.32$	$= 0.0026$	$\sigma_3 = 0.017$
Heterogeneous mixture	Normal-logistic		$= 0.0049$	$= 0.0117$
			$= 0.0017$	$= 0.0069$

Interestingly, the results obtained indicate that the performance of Gaussian mixtures is better than that of heavy-tailed distributions, reflecting the flexibility of the Gaussian mixture. This model also describes the high kurtosis values with finite component Gaussian mixture model. Additionally, the sum of the rank of information criteria and log-likelihoods for the Gaussian mixture is lower than that of the previously used fat-tail models. Table 1 and Table 2 present the estimated parameters of best fitted models

Table 2. Estimated parameters for best-fit NIG, Gaussian mixtures and heterogeneous mixture models implemented on PKR-USD exchange rate returns.

Model Type	Best-Fit Model	Parameters		
Heavy-tailed distributions	NIG	$\mu = 7.59 \times 10^{-5}$		
		$\delta = 3.23 \times 10^{-3}$		
		$\alpha = 36.47$		
		$\beta = 2.43$		
Gaussian mixture	Three components	$w_1 = 0.59$	$\mu_1 = 3.42 \times 10^{-4}$	$\sigma_1 = 9.11 \times 10^{-3}$
		$w_2 = 0.37$	$\mu_2 = 4.51 \times 10^{-5}$	$\sigma_2 = 1.17 \times 10^{-3}$
		$w_3 = 0.04$	$\mu_3 = 1.88 \times 10^{-3}$	$\sigma_3 = 0.023$
Heterogeneous mixture	Normal-logistic	$w_1 = 0.36$	$\mu_1 = 4.11 \times 10^{-5}$	$\sigma_1 = 1.14 \times 10^{-3}$
		$w_2 = 0.64$	$\mu_2 = 4.12 \times 10^{-4}$	$\sigma_2 = 5.66 \times 10^{-3}$

Our last approach employs heterogeneous mixtures to offer more flexibility due to the different nature of the component distributions. Following the ranks-based model selection, our results suggest that the Normal-logistic-mixture is the best model for describing the returns of both stock and currency returns. The best fitted Normal-logistic-mixture model suggests that tails of returns deviate significantly from light-tailed distribution. The proposed mixture model effectively captures medium to high peaked distributions. However, one of the disadvantages of heterogeneous mixtures is the computational difficulty to measure the VaR. On the other hand, density plots are a simple and effective tool to analyze the goodness of the selection of best-fit distribution. It visually summarizes the Goodness-of-fit and presents a comparison between implemented distributions. Figure 3 shows the density plots of PSX-100 (left) and PKR-USD (right) returns for three-component Gaussian mixtures models, which is the best-fit distribution among heavy-tailed distributions, Gaussian mixtures up to three components and heterogeneous-mixture models.

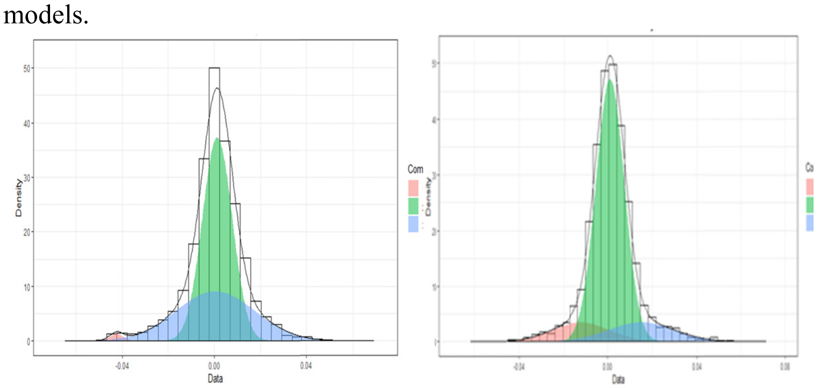

Figure 3. Density plot of best-fit model, three component Gaussian mixture, implemented on PSX-100 (left) and PKR-USD (right) returns

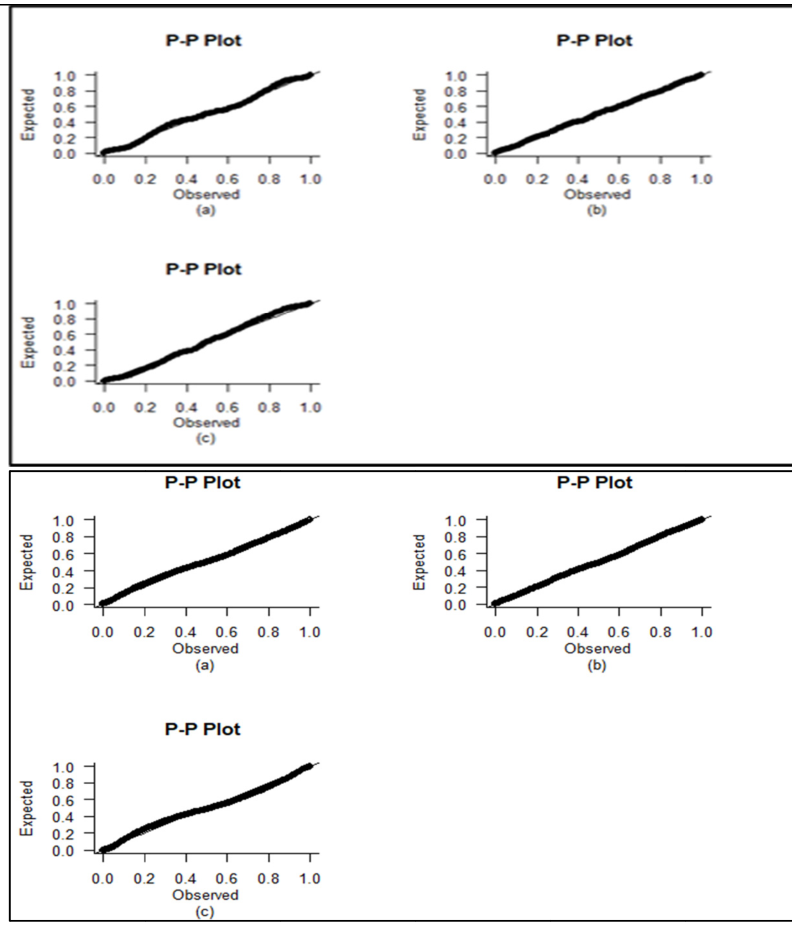

Figure 4. P-P plot of best-fit models implemented on PSX-100(top) and PKR-USD (bottom) returns, for both time-series: (a) NIG distribution (b) Three component Gaussian model (c) Normal-Logistic mixture model

Density plots are sensitive to bandwidth size and can distort the data due to under or over-smoothing. To assess the goodness-of-fit in detail, we have further considered P-P plots. Figure 4 represents the P-P plots of the PSX-100 (top) returns and PKR-USD (bottom) exchange rate returns. All plots show the best fit, with no significant deviation. Comparatively heterogeneous models show low non-conformity at the tails, which is more prominent on the left tail of the graph. Likewise, heavy-tailed distributions lag Gaussian mixture models as divergence is slightly visible at the left tail. Gaussian mixture best captures the returns from all aspects in the area around means and tails. Unlike P-P plots of PSX-100 returns, mild deviations are observed in the PKR-USD and indicate the volatile nature of the currency market compared to the equity market. Fat-tailed distribution and heterogonous mixture show low deviations around the mean and at the tails. Overall, Gaussian mixtures with three components still outperform other models.

4.2. Volatility Modelling and Forecasting Performance of PSX-100 and PKR-USD Exchange Rates

In this section, we investigate the estimation and prediction of volatility by employing traditional GARCH models. We present empirical results of fitted SGARCH (1,1), IGARCH (1,1), EGARCH (1,1), GJR-GARCH (1,1), APARCH (1,1), and CSGARCH (1,1) models to study the leverage effect, impact of good and bad news on the volatility series of underlying data sets, asymmetric nature of returns. Conditional probability distributions used in GARCH models are student-T, skewed Student-T, generalized hyperbolic, skewed-generalized error distribution and Normal inverse Gaussian. We have considered log-likelihood and AIC, BIC, and HQC to jointly determine the best-fit GARCH model using the minimum average rank-based approach. We observe the EGARCH (1,1) model with Student-T distribution best fits the PSX-100 returns. The PKR-USD returns follow the APARCH (1,1) model with skewed Student-T distribution of residuals. The EGARCH (1,1) and APARCH (1,1) models show the minimum ranks equal to 1, respectively. Our model selection approach follows the ranks computation of all underlying information criteria and log-likelihood values. Further, for each GARCH model, the mean and median values of computed ranks decide the best-fitted model.

Table 3. GARCH models estimation results for PSX-100 returns

PSX-100 Returns						
Parameters	**SGARCH**	**IGARCH**	**GJR-GARCH**	**EGARCH**	**APARCH**	**CSGARCH**
μ	0.001013 (0.000809)	0.001012 (0.000299)	0.000814 (0.000506)	0.000460 (0.000394)	-0.000009 (0.000006)	0.001025 (0.000522)
$AR1$	0.157947 (0.239849)	0.158341 (0.204415)	0.294917 (0.240685)	0.313240 (0.048184)	0.671999 (0.010762)	0.142777 (0.186516)
$MA1$	-0.023923 (0.237345)	-0.024283 (`0.205478)	-0.148697 (0.243647)	-0.169970 (0.044429)	-0.546870 (0.028274)	-0.007974 (0.187866)
ω	0.000004 (0.000017)	0.000004 (0.000006)	0.000006 (0.000014)	-0.486090 (0.078425)	0.000001 (0.000009)	0.000004 (0.000014)
α_1	0.203649 (0.063846)	0.204398 (0.058366)	0.107383 (0.090038)	-0.113430 (0.018981)	0.212170 (0.032131)	0.045852 (0.091718)
β_1	0.795348 (0.138859)	0.795602 (0.128765)	0.770874 (0.098278)	0.946380 (0.008376)	0.846934 (0.023965)	0.559038 (0.394713)
λ_1	5.240917 (0.539177)	5.224560 (0.684703)	5.411278 (1.378847)	5.347320 (0.648172)	4.116681 (0.395551)	5.343978 (0.569972)
γ_1			0.228447 (0.110941)	0.337320 (0.031439)	0.295692 (0.029569)	
δ					0.703764 (0.703764)	
η_1						0.993919 (0.034161)
η_2						0.179238 (0.137426)

For PSX-100 returns, our empirical findings suggest that α_1 and β_1 are significant at a 1% level with a sum less than one, which implies that returns are stationary. Reaction parameter α_1 is less than 0.05 and indicates stable market returns. The parameter β_1 is significant and suggest the presence of volatility clustering. Volatility clustering implies that periods of low volatility are followed by high volatility and vice-versa. Value of β_1 parameter is more than 90%, suggesting that volatility exhibit a highly persistent behavior. Overall, Ljung-Box statistics for residuals and square residuals are significant confirming the fact that EGARCH (1,1) serves as the best-fit GARCH model and captures the asymmetric effect of the PSX-100 market. Also, an asymmetry exists if the parameter $\gamma_1 > 0$. We can see from Table 3 the value of $\gamma_1 = 0.337320$.

Table 4. GARCH models estimation results for PKR-USD exchange rate returns

	PKR/USD					
	SGARCH	**IGARCH**	**GJR-GARCH**	**EGARCH**	**APARCH**	**CSGARCH**
μ	0.000029 (0.00015)	0.000029 (0.00005)	0.000029 (0.00014)	-0.000012 (0.00003)	0.000014 (0.000015)	0.000074 (0.007678)
$AR1$	-0.013636 (0.064882)	-0.014168 (0.044151)	-0.013403 (0.07538)	-0.019438 (0.006305)	-0.021613 (0.002372)	-0.012166 (5.671148)
$MA1$	-0.543341 (0.031477)	-0.542864 (0.023731)	-0.547418 (0.058801)	-0.551305 (0.027288)	-0.544112 (0.015304)	-0.522827 (5.884535)
ω	0.000000 (0.000058)	0.000000 (0.000037)	0.000000 (0.000062)	-0.220703 (0.058633)	0.000000 (0.000004)	0.000000 (0.00031)
α_1	0.166551 (1.290337)	0.169356 (0.866097)	0.175274 (1.342186)	-0.003135 (0.023605)	0.285991 (0.045464)	0.063945 (2.099376)
β_1	0.832449 (1.420273)	0.830644 (1.410345)	0.833005 (1.528836)	0.976646 (0.005752)	0.820554 (0.031345)	0.694755 (0.87449)
λ_1	4.322853 8.758358	4.301532 6.620937	4.348178 9.587804	2.737940 (0.261530)	2.843719 0.088492	4.875147 0.280104
λ_2	0.852542 0.016197	0.852243 0.019176	0.852762 0.060320	0.817712 (0.026694)	0.834970 0.022047	0.857510 0.899362
γ_1			-0.019994 0.092886	0.500113 (0.065823)	0.012537 0.042788	
δ					0.826407 0.092845	
η_1						0.989570 (0.807016)
η_2						0.061917 (0.025157)

Note: Values in parenthesis indicate standard errors. $\boldsymbol{Q(9)}$ and $\boldsymbol{Q^2(9)}$ respectively represent Ljung statistics of order 9 for residuals and square residuals

For PKR-USD exchange rate returns, APARCH (1,1) with skewed Student-T distribution is the best-fit GARCH model. Table 4 presents estimated parameters of the best fit GARCH model for PKR-USD exchange rate returns along with diagnostic test results. APARCH (1,1) with skewed Student-T distributions is the best performing GARCH model. Both α_1 and β_1 are significant at 1% level of significance. The significantly high value of parameter α_1 represents high uncertainty and low investor confidence prevailing in currency markets. The value of β_1 parameter is more than 80%, suggesting that volatility exhibit a highly persistent behavior. The skewness coefficient λ_2 for skewed Student-T distribution is significant and suggests that returns are positively skewed. Therefore, an asymmetry exists between losses and gains in PKR-USD exchange rate. The reaction parameter δ of volatility for negative shocks is equal to 0.826, which is more than the positive shocks. It shows the asymmetric nature of PKR currency returns. Consequently, negative shocks cause a sudden decline followed by a slow recovery. The large value of the shape parameter λ_1 also suggests that high volatiles periods take a longer time to die out.

The forecasting performance of GARCH models are assessed through loss functions, which help to compare the out-of-sample forecasting accuracy of the model for a window of 22 days. We have considered three loss functions:(i) mean-square error (MSE), (ii) mean-absolute error (MAE) and (iii) mean-absolute-percentage error (MAPE).

Table 5. Ranks for forecasting performance for PSX-100 and PKR-USD exchange rate returns

Student-T Distributions for PSX-100 returns				
	MSE	MAE	MAPE	Mean Rank
SGARCH	1	4	4	3.0
IGARCH	3	5	6	4.7
GJRGARCH	4	3	3	3.3
EGARCH	2	2	2	2.0
APARCH	6	6	1	4.3
CSGARCH	5	1	5	3.7
Skewed Student-T Distributions for PKR-USD returns				
	MSE	MAE	MAPE	Mean Rank
SGARCH	4	1	6	3.7
IGARCH	3	2	5	3.3
GJRGARCH	6	3	2	3.7
EGARCH	1	5	4	3.3
APARCH	2	4	3	3.0
CSGARCH	5	6	1	4.0

Table 5 shows that EGARCH (1,1) model with Student-T distribution and APARCH (1,1) with skewed Student-T are best-fitted models with mean information criteria ranks of 2 and 3 for PSX-100 and PKR-USD returns, respectively. The minimum average rank-based approach observed that EGARCH (1,1) and APARCH (1,1) models are respectively the best models to forecast the volatilities of the two time-series.

4.3. Value-at-Risk and Detrended Correlation Coefficient analysis for PSX-100 vs. PKR-USD returns

Value-at-Risk is one of the most widely used one-sided risk measures effective in forecast evaluation, countering asymmetric response of volatility measures towards outliers, and in unevenly distributed observations. Figure 5 presents the Value-at-risk of the best-fit GARCH model. It shows that PSX-100 is much more volatile than the PKR-USD forex market. Returns in the currency market experience small fluctuations compared to the stock market. However, currency returns have significantly persistent volatility. Also, the leverage effect is observed in currency return, i.e., volatility increases when returns drop down, and vice versa. Overall, PSX-100 returns are more volatile than PKR-USD exchange rate returns. As markets are getting more integrated, volatility transmission and linkage between markets received a lot of research attention. Flow oriented model (Dornbusch & Fischer, 1980) and stock-oriented model (Branson and Henderson 1985; Frankel 1983) are two major frameworks to understand this linkage. Flow oriented model proposes a positive relationship between stock and currency returns. On the contrary, the stock-oriented model suggests a negative correlation. We use Detrended Correlation Coefficient Analysis (DCCA) to analyse the empirical relationship. Figure 5 (right) represents the co-movements between these variables. All correlations are negative, and the degree of correlation is increasing with window length. Hence, the stock-oriented model is more pertinent in the relationship between PSX-100 and PKR-USD exchange rate.

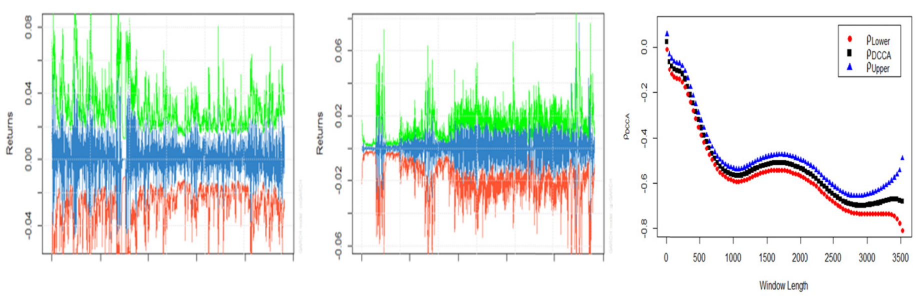

Figure 5: Value-at-Risk plots at 1% VaR limit for PSX-100 (left) and PKR-USD (center) returns, and Detrended Correlation Coefficient analysis for PSX 100 vs. PKR-USD (right) returns

5. Conclusions

In the first step, three approaches have been used to find the most suitable return distribution model of PASX-100 and the PKR-USD data series. The first approach follows seven different univariate returns distributions such as Logistic, Laplace, Student-T, Skew Generalized. T, NIG, Generalized Hyperbolic, and Variance-Gamma. The results of negative log-likelihood and five different information criteria suggest NIG a best-fitted model of returns distribution for both the stock and currency market. Also, density plots and PP plots are captured well in the case of NIG model. The second approach follows employment of the Gaussian mixture model. We examined a three-component Gaussian mixture that is suitable and flexible to capture the high kurtosis of both returns of PSX-100 and PKR-USD. The estimated parameters of these models are simple and easy to elaborate on with economic significance. These parameters such as weights, the mean and standard deviation explain the factors affecting the underlying stock market. Additionally, we present the first study that follows heterogeneous mixtures for modelling PSX-100 and PKR-USD returns. This approach follows a heterogeneous mixture model such as Normal-Logistic, Normal-Laplace, Logistic-Laplace, Normal-NIG, Logistic-NIG, Laplace-NIG, and Logistic-Skewed-Generalized-T distribution. We propose these mixtures capture effectively medium to high peaked returns. In this study, the Normal-Logistic mixture model is most appropriate for both asset returns.

In the third step, we employed the most widely used traditional technique of GARCH modelling to estimate and forecast the volatilities of PSX-100 and PKR-USD returns series. For this purpose, we choose univariate conditional volatility models, namely SGARCH (1,1), IGARCH (1,1), GJR-GARCH (1,1), EGARCH (1,1), APARCH (1,1), and CSGARCH (1,1). These models have been significantly used to estimate and forecast the volatilities of currency and stock returns. Also, these models capture asymmetry and leverage effect, which is the negative correlation between returns shocks and consequent volatility shocks, and persistence of the volatility. GARCH models suggest the presence of a strong leverage effect and volatility clustering in each market. Both markets exhibit persistent volatilities in nature, which is a primary characteristic of the emerging market. We used log-likelihood, AIC, BIC, CAIC, AICC and HQC information criteria to find the best suitable volatility estimation model. Each selection criteria have its advantages and disadvantages based on the number of parameters and sample size. Therefore, we evaluated the rank of each GARCH model by using all the available information of penalized selection criteria and log-likelihood values. The model with the least rank is considered as best-fit model. We conclude PSX-100 returns follows the EGARCH (1,1) model with Student-T distribution. Similarly, the PKR exchange rate returns in USD follows the APARCH (1,1)-skewed Student-T distribution with the minimum rank of 3. Both best-fitted models suggested the existence of the leverage effect in the PSX-100 and PKR-USD. A high variation in the estimated volatilities of both assets has observed and behaviour of the variance is asymmetric for both series.

Finally, value-at-risk analysis implies that investments in the Pakistan Stock exchange are subject to more market risk as compared to the Pakistani Rupee. Similarly, the volatility of PSX-100 is comparatively more persistent. Thus, returns of PKR-USD exchange rates are relatively a safer investment. In the last step, we performed a DCCA analysis that suggests the use of a stock-oriented model i.e., equity and currency markets are negatively correlated. Therefore, both assets can be hedged against each other to limit the portfolio risk. These findings have significant implications for investment decision making in these high-risk markets.

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Appendix A

GARCH Model	Mathematical Expression	Parameter
SGARCH (Bollerslev 1986)	$\sigma_t^2 = \omega + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2$ for $t \in \mathbb{Z}$	$\omega \geq 0, \alpha_1, \beta_1 \geq 0$ $\sigma_t^2 > 0 \; \forall t$ Volatility persistence parameter: $\alpha_1 + \beta_1$ Weak stationairty condition, $\alpha_1 + \beta_1 < 1$
APARCH (Ding et al. 1993)	$\sigma_t^\delta = \omega + \alpha_1(\lvert e_{t-1}\rvert - \gamma_1 e_{t-1})^\delta + \beta_1 \sigma_{t-1}^\delta$	$\delta > 0, \lvert\gamma_1\rvert \leq 1$ Leverage parameter, $\lvert\lambda_1\rvert \leq 1$
GJR-GARCH (Glosten et al. 1993)	$\sigma_t^2 = \omega + (\alpha_1 + \gamma_1 I_{t-1}) e_{t-1}^2 + \beta_1 \sigma_{t-1}^2$	$\alpha_1 > 0, \beta_1 > 0, \gamma_1 > 0, \omega > 0$ Asymmetry parameter, γ_1
EGARCH (Nelson 1991)	$\log(\sigma_t^2) = \omega + \alpha_1 \left(\lvert Z_{t-1}\rvert - \mathbb{E}(\lvert Z_{t-1}\rvert)\right) +$ $+ \beta_1 \log(\sigma_{t-1}^2) + \gamma_1 Z_{t-1}$	$\delta > 0, \alpha_1 \geq 0, \; \beta_1 \geq 0, \lvert\gamma_1\rvert < 1, \omega > 0$ Sign parameter, α_1 Leverage parameter, γ_1 Stationarity condition, $0 < \beta_1 < 1$
CSGARCH (Lee and Engle 1999)	$\sigma_t^2 = q_t + \alpha_1(e_{t-1}^2 - q_{t-1}) + \beta_1(\sigma_{t-1}^2 - q_{t-1})$ $q_t = \omega + \rho q_{t-1} + \phi(e_{t-1}^2 - \sigma_{t-1}^2)$	$\delta > 0, \alpha_1 \geq 0, \beta_1 \geq 0, \phi \geq 0, \omega \geq 0$ Persistence condition, $\rho < 1$

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